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LETTER TO THE EDITOR

Universality classes of critical antiferromagnets

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Abstract. We derive the possible values of the conformal anomaly number c for the integrable Heisenberg antiferromagnet with planar anisotropy $\gamma = \pi/\nu$ $(0 < \gamma \leqslant \pi/2)$ in the case where ν is a rational number. For given spin S, with $\nu > 2S$, we find $c = 3S/(S+1)$, suggesting a renormalisation onto the $k = 2S$ Wess–Zumino–Witten fixed point. In contrast, for $\nu \leqslant 2S$ a new renormalisation behaviour is revealed, with $2S - 1$ distinct universality classes indexed by the integer part of ν.

Recent advances in conformal field theory have made possible a new attack on the longstanding problem of understanding the critical behaviour of antiferromagnetic quantum spin chains. As is well known, the strong quantum fluctuations present at the critical point, $T = 0$, make this a very subtle problem. While a rather complete theory exists for spin-$\frac{1}{2}$ due to the pioneering work by Luther and Peschel (1975), independently supported by analytical (Haldane 1980, Izergin and Korepin 1985) as well as numerical calculations (see Moreo (1987) and references therein), it is only now that a picture for the higher-spin models is beginning to emerge (Affleck 1985, 1986a, b, Affleck and Haldane 1987, Affleck *et al* 1988). In the *isotropic* case all critical theories are given by the SU(2) Wess–Zumino–Witten (WZW) models with topological coupling k, a positive integer. The stable fixed point is $k = 1$ (Schulz 1986, Ziman and Schulz 1987), corresponding to a free massless boson, which attracts a large set of half-odd-integer spin models, including those with pure exchange interactions (minimal models). In contrast, integer-spin Hamiltonians generically exhibit non-critical behaviour, the criticality being suppressed by the collective excitations acquiring a mass (Haldane 1983a, b). The higher-k theories, on the other hand, represent multicritical points in the space of spin interactions, with the *integrable* spin-S Hamiltonians (Takhtajan 1982, Babujian 1983, Johannesson 1986) being attracted to the $k = 2S$ multicritical point, i.e. the central charge (conformal anomaly number) of the underlying Virasoro algebra is here given by $c = 3S/(S+1)$.

Of obvious interest is to understand how to pass to the case of anisotropic spin interactions, i.e. when the global SU(2) symmetry is broken down to U(1). Not only do most one-dimensional magnets in the laboratory exhibit anisotropies, but the most interesting theoretical applications of spin-chain physics, to quantum field theory and many-particle problems in general, typically require the introduction of an anisotropy.

In this case one expects $c = 1$ at criticality, i.e renormalisation onto a free boson (Affleck 1985). It was recently shown, however, (Johannesson 1988) that the conformal anomaly number of the integrable higher-spin Heisenberg antiferromagnet with planar anisotropy $\gamma = \pi/\nu$ (Sogo 1984, Kirillov and Reshetikin 1985, 1987a, b, Babujian and

Tsvelick 1986) remains at the value $c=3S(S+1)$ when ν is an integer greater than $2S$. In other words, the introduction of an anisotropy of this type does *not* destabilise the $k=2S$ multicritical point. This result is rather remarkable since it implies that the fluctuations of the WZW fields remain massless despite the chiral symmetry of the theory having been lowered from SU(2) to U(1). This suggests that some hidden symmetry is now protecting the massless sector.

In this letter we wish to extend our analysis to the case where ν is a rational number greater than or equal to 2, i.e. with $0<\gamma\leqslant\pi/2$. While for $\nu>2S$ we recover the result obtained for integer ν, a novel phenomenon is encountered when $\nu<2S$: the conformal anomaly is now indexed by the parameter $[\nu]$, where $[\nu]$ is the largest integer less than or equal to ν. Hence, in this case there is a variety of possible renormalisation fixed points which can be made attractive by proper tuning of the anisotropy. Put differently, the types of relevant operators produced under renormalisation now depend on how the value for the anisotropy parameter has been chosen. Unfortunately, the mechanism which gives rise to this unusual behaviour seems rather elusive.

We begin by defining the Hamiltonian. Let V_j be a copy of C^{2S+1} with $j=1,\ldots,N$, $V_1=V_{N+1}$. Then (Kirillov and Reshetikin 1985)

$$H_S=-\tfrac{1}{2}\mathrm{i}\gamma\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{j=1}^{N}P_{j,j+1}R_{j,j+1}(2S|u,\gamma)\right.$$
$$\left.-\mathrm{i}N\prod_{j=2S-1}^{2S}\sinh(u+\mathrm{i}\gamma j)\prod_{k=0}^{2j-1}\sinh[u+\mathrm{i}\gamma(2S+k)]\right)\qquad u=-2\mathrm{i}\gamma S \tag{1}$$

where $P_{j,j+1}$ is the exchange operator in $V_j\otimes V_{j+1}$ and $R_{j,j+1}$ is a linear operator acting in the same space, obtained by the standard fusion procedure (Kulish *et al* 1981) from the elementary Baxter bundles

$$R_{j,j+1}(1|u,\gamma)=\sinh[u+\tfrac{1}{2}\mathrm{i}\gamma(1+\sigma_j^z\sigma_{j+1}^z)]+\mathrm{i}\sin\gamma\,(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+). \tag{2}$$

Here $\sigma^\pm=\sigma^x\pm\mathrm{i}\sigma^y$, σ^z are the Pauli matrices. By choosing $S=\frac{1}{2}$ in (1), the usual *XXZ* model is recovered:

$$H_{1/2}=\frac{\gamma}{4\sin\gamma}\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cos\gamma\,\sigma_j^z\sigma_{j+1}^z\qquad \boldsymbol{\sigma}_1=\boldsymbol{\sigma}_{N+1} \tag{3}$$

where the prefactor is due to the normalisation in (1). Choosing $S=1$ gives (Zamolodchikov and Fateev 1980)

$$H_1=\frac{\gamma}{2\sin(2\gamma)}\sum_{j=1}^{N}\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}-(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})^2-2(\cos\gamma-1)$$
$$\times[(S_j^xS_{j+1}^x+S_j^yS_{j+1}^y)S_j^zS_{j+1}^z+S_j^zS_{j+1}^z(S_j^xS_{j+1}^x-S_j^yS_{j+1}^y)]$$
$$+2\sin^2\gamma\,(1-S_j^zS_{j+1}^z)S_j^zS_{j+1}^z+4\sin^2\gamma\,(S_j^z)^2\qquad \boldsymbol{S}_1=\boldsymbol{S}_{N+1} \tag{4}$$

where S^x, S^y, and S^z are the spin-1 operators in C^3. As already mentioned, we shall require that $0<\gamma\leqslant\pi/2$.

It might be worthwhile pointing out that, despite its formidable appearance, the Hamiltonian in (1) defines an extremely simple dynamics, distinguished by exhibiting *no* mixing of the momentum distribution, as follows from its complete integrability.

The model can be diagonalised exactly by a Bethe ansatz (Sogo 1984, Kirillov and Reshetikin 1985, Babujian and Tsvelick 1986) and one finds, with the normalisation in (1), a spectrum

$$E(\lambda_1,\ldots,\lambda_M)=\tfrac{1}{2}\gamma\sum_{k=1}^{M}\frac{\sin(2S\gamma)}{\sinh[(\gamma/2)(\lambda_k+2\mathrm{i}S)]\sinh[(\gamma/2)(\lambda_k-2\mathrm{i}S)]} \tag{5}$$

with associated momenta

$$p(\lambda_1,\ldots,\lambda_M)=\sum_{k=1}^{M}2\tan^{-1}[\tanh(\gamma\lambda_k/2)\cot(\gamma S)] \tag{6}$$

where the parameters $\lambda_1,\ldots,\lambda_M$ satisfy the coupled equations

$$\left(\frac{\sinh[(\gamma/2)(\lambda_j+2\mathrm{i}S)]}{\sinh[(\gamma/2)(\lambda_j-2\mathrm{i}S)]}\right)^N=-\prod_{k=1}^{M}\frac{\sinh[(\gamma/2)(\lambda_j-\lambda_k+2\mathrm{i})]}{\sinh[(\gamma/2)(\lambda_j-\lambda_k-2\mathrm{i})]}\qquad j=1,\ldots,M \tag{7}$$

for some integer M.

In the limit $N\to\infty$, with M/N fixed, the solutions of (7) cluster into strings in the complex plane

$$\lambda^n_{\alpha,j}=\lambda^n_\alpha+\mathrm{i}(n+1-2j+\tfrac{1}{2}(1-v_{2S}v_n)[\nu])+\mathrm{O}(\mathrm{e}^{-\delta N})\qquad \delta>0\qquad j=1,\ldots,n \tag{8}$$

where $\lambda^n_\alpha=\mathrm{Re}(\lambda^n_{\alpha,j})$, and v_{2S} and v_n are spin parities taking the values ±1. To classify the allowed string configurations one defines a set of integers y_i and m_i (Takahashi and Suzuki 1972):

$$y_{-1}=0\qquad y_0=1\qquad y_1=b_0\qquad y_{i+1}=y_{i-1}+b_iy_i\qquad i\geqslant0 \tag{9a}$$

$$m_0=0\qquad m_1=b_0\qquad m_{i+1}=m_i+b_i\qquad i\geqslant0 \tag{9b}$$

with the numbers b_i being the elements in the continued fraction expansion of $\nu=\pi/\gamma$, i.e.

$$\nu=[b_0,b_1,b_2,\ldots]\equiv b_0+\cfrac{1}{b_1+\cfrac{1}{b_2+\cdots}}. \tag{10}$$

It is here also useful to introduce the real numbers p_i:

$$p_0=\nu\qquad p_i/p_{i+1}=[b_i,b_{i+1},\ldots]\qquad i\geqslant0. \tag{11}$$

One can now prove that the number of elements n_j and parity v_j of a string are given by

$$n_j=y_{i-1}+(j-m_i)y_i\qquad m_i\leqslant j<m_{i+1} \tag{12}$$

and

$$v_{n_1}=1\qquad v_{m_1}=-1\qquad v_{n_j}=\exp\{\mathrm{i}\pi[(n_j-1)/\gamma]\}\qquad j\neq n_1,m_1 \tag{13}$$

respectively, where again $[x]$ denotes the integer part of x. In addition, when $S>\frac{1}{2}$, one must require the existence of an integer r such that for some k

$$1+2S=n_k\qquad m_r\leqslant k<m_{r+1}. \tag{14}$$

As shown by Kirillov and Reshetikin (1987a), when $r=0$ or 1 the vacuum is built from strings of one type only, allowing a rather easy access to various static properties of the model. In particular, restricting ν to a rational number we find for the low-temperature asymptotics of the free-energy per unit length

$$f=\begin{cases}\text{constant}-p_1\dfrac{S}{S+1}T^2+\mathrm{O}(T^3) & \nu>2S \qquad\qquad\qquad\qquad\qquad r=0\\[2ex] \text{constant}-p_2\dfrac{[\nu]}{[\nu]+2}T^2+\mathrm{O}(T^3) & 2\leqslant\nu\leqslant 2S<b_1[\nu];\ b_1\geqslant 2;\ S\geqslant 1 \qquad r=1.\end{cases} \tag{15}$$

This result allows for a computation of the conformal anomaly number c, which labels the possible universality classes at $T=0$. As realised by Affleck (1986b) (see also Blöte *et al* 1986), by integrating the trace anomaly relation for a 2D massless field theory over an infinitely long cylinder of circumference $1/T$, a conformal mapping of the cylinder onto the unit disc reveals a universal finite-size correction to the free-energy per volume:

$$\lim_{L\to\infty} F/L=\text{constant}-\tfrac{1}{6}\pi c T^2. \tag{16}$$

Hence, by normalising the velocities of the collective excitations in the spin model to unity, assuming them to be represented by the 2D relativistic theory (now interpreted as a finite-temperature theory in the infinite plane) the conformal anomaly can be read off from (15) and (16) by identifying terms quadratic in T.

The energy-momentum relation in the low-energy limit is given by (Kirillov and Reshetikin 1987a)

$$E(p)=\begin{cases}p\pi/2p_1 & r=0\\ p\pi/2p_2 & r=1\end{cases} \tag{17}$$

which yields a velocity

$$v=\begin{cases}\pi/2p_1 & r=0\\ \pi/2p_2 & r=1.\end{cases} \tag{18}$$

Normalising v to unity implies that (15) will be multiplied by a factor v. Comparison with (16) then gives for the conformal anomaly

$$c=3k/(k+2) \tag{19}$$

where

$$k=\begin{cases}2S & \nu>2S\\ [\nu] & 2\leqslant\nu\leqslant 2S<b_1[\nu];\ b_1\geqslant 2;\ S\geqslant 1.\end{cases} \tag{20}$$

When $\nu>2S$ we thus recover the result reported in Johannesson (1988) for integer ν. In particular, this gives some added weight to a recent conjecture by Alcaraz and Martins (1988) that under renormalisation the $S=1$ Hamiltonian (4) flows towards the $k=2$ WZW fixed point for *all* values of γ in the range $0\leqslant\gamma\leqslant\pi/2$. On the other hand, in the interval $2\leqslant\nu\leqslant 2S$ we find a new behaviour. There are here $2S-1$ possible values of c, explicitly depending on the choice of ν, and hence $2S-1$ distinct universality classes. Since $[\nu]$ is an integer, one is led to expect renormalisation onto one of $2S-1$ distinct WZW fixed points, corresponding to the possible values of the topological coupling $k=[\nu]$, $[\nu]=2,\ldots,2S$. To check this, one should identify the relevant and marginal operators of a WZW model with broken SU(2) symmetry and then make a comparison with the operator content of the spin problem. If the test reveals different scaling dimensions, we are instead dealing with a collection of new universality classes. Either way, the scenario is quite intriguing. (We should perhaps add that performing the suggested test will be no easy matter, considering the current state of available methods.)

Why should one expect the renormalisation behaviour to depend on a relation between spin and anisotropy? As follows from our analysis, when $\nu > 2S$ the renormalisation is fixed by the *spin* only, while for $\nu < 2S$, with $S > 1$, the choice of *anisotropy* decides which of several possible fixed points will attract the Hamiltonian. Before attempting an answer, it might be important to note that other properties of the model are also connected to a relation between anisotropy and spin. As observed by Kirillov and Reshetikin (1987a), all diagonal spin operators $S_j^z, j = 1, \ldots, N$, are present in the Hamiltonian only through algebraic combinations of $\exp(\mathrm{i}\gamma S_j^z)$. There are thus two periods characterising the model. One is given by the spin value through U(1) rotations, while the other is determined by the presence of the $\exp(\mathrm{i}\gamma S_j^z)$ terms in the Hamiltonian and hence is equal to 2ν. If $\nu > 2S$, the values of $\exp(\mathrm{i}\gamma S_j^z)$ are restricted to a semicircle as S_j^z runs through its allowed values $-S, \ldots, S$. On the other hand, if $\nu < 2S$ there is no such constraint on the possible values of $\exp(\mathrm{i}\gamma S_j^z)$. As found by Kirillov and Reshetikin, the structure of the vacuum differs fundamentally in the two cases†. The two distinct types of critical behaviour found here thus coincide with the appearance of two different kinds of vacua. However, it remains to be established if, and how, this fact explains the observed renormalisation.

A detailed account of the results presented here will be given in a later work.

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† In the concluding paragraph of Kirillov and Reshetikin (1987a) it is stated that two distinct vacua occur for $\nu < S$ and $\nu > S$, respectively. However, as should be evident from the preceding analysis, this is a misprint (N Yu Reshetikin, private communication).

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